Ring Theory 5

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Q) A ring R has infinitely many nilpotent elements if $a, b \in R$ if $ba \neq 1$ and $ab = 1$.

Ans:- $ab - 1 = 0$

$\Rightarrow ab - 1 \in$ Nil radical of R

$R(ab-1) \in$ Nil radical of R as it is an ideal

$\Rightarrow$ R has infinitely many nilpotent elements

<u>Definition</u>:- When I is an ideal of ring R then R/I with binary operations as $(r+I) + (s+I) = (r+s) + I$ and $(r+I) \times (s+I) = rs + I \quad \forall r, s \in R$

is called the quotient ring of R by I

<u>First Isomorphism Theorem of Rings</u>:-

If $\phi : R \to S$ is a homomorphism of rings then the kernel of ϕ is an ideal of R. The image of ϕ is a subring of S and $R/\ker(\phi)$ is isomorphic as a ring to $\phi(R)$

<u>Proof</u>:- $\ker(\phi) = \{ r : \phi(r) = 0 \} = I$ (let)

$r + I$ are the cosets for $r \in R$

$\phi(r_1)\phi(r_2) = \phi(r_1 r_2)$

$\phi(r_1 r_2) = r_1 r_2 + I$

$\phi(r_1) = (r_1 + I) \quad \phi(r_2) = (r_2 + I)$

$(r_1 + I)(r_2 + I) = (r_1 r_2 + I)$

$\phi(r_1) = (r_1 + I)$

$(r_1 + I)(r_2 + I) = (r_1 r_2 + I)$

I is a subring of R

$r_1 r_2 + r_1 I + r_2 I + I = r_1 r_2 + I$

$(r_1 + r_2) I + I = I$

$r_3 I + I = I$

$\hookrightarrow r_3 I \subseteq I$

$\hookrightarrow r_3$ is arbitary

So I is an ideal

R/I is a ring, $f : R \to R/I$, $r \to r + I$ is a group homomorphism with Kernel I

$f(rs) = rs + I = (r+I)(s+I) = f(r) f(s)$

So f is a ring homomorphism.

$g(f)$ will be ϕ

$g : R/I \to \phi(R)$

$r + I \to \phi(r)$

$\hookrightarrow$ It is a bijection & homomophic.

So R/I is isomorphic as ring to $\phi(R)$

Q> Suppose R and S are rings and $\phi : R \to S$ is a ring homomorphism. e is an idempotent in ring R. Prove that $\phi(e)$ is idempotent in ring S.

Ans:- $e^2 = e$

$\phi(e)^2 = \phi(e)\phi(e) = \phi(ee) = \phi(e^2) = \phi(e)$

Q> Define an example of an injective ring homomorphism from $\mathbb{Z}/\_\_\mathbb{Z}$ to $\mathbb{Z}/20\mathbb{Z}$.

Q> Define an example of ...
$\mathbb{Z}/5\mathbb{Z}$ to $\mathbb{Z}/20\mathbb{Z}$.

Ans:- $\mathbb{Z}/20\mathbb{Z}$ has a unique subgroup of order 5 which is generated by $4+20\mathbb{Z}$.

$16+20\mathbb{Z} \subset \langle 4+20\mathbb{Z} \rangle$

$(16+20\mathbb{Z})^2 = 256 + 20\mathbb{Z} = 16 + 20\mathbb{Z}$ → so it is an idempotent

$$\phi : \mathbb{Z} \to \mathbb{Z}/20\mathbb{Z}$$
$$n \to n(16+20\mathbb{Z}) \quad \forall n \in \mathbb{Z}$$

$\ker(\phi) = 5\mathbb{Z}$

By using First Isomorphism Theorem:-

$f : \mathbb{Z}/5\mathbb{Z} \to \mathbb{Z}/20\mathbb{Z}$ is a homomorphism

$$f(n+5\mathbb{Z}) = \phi(n) = n(16+20\mathbb{Z}) = 16n + 20\mathbb{Z}$$

Second Isomorphism Theorem:-

Let R be a ring, S be a subring of R and I an ideal of R. Then

(1) The sum $S+I = \{s+i : s \in S, i \in I\}$ is a subring of R

(2) $S \cap I$ is an ideal of S

(3) $(S+I)/I \cong S/(S\cap I)$

Proof:- (1) $S \subseteq R$ and I is ideal, so we get $1 \in S+I$

Let s_1+a_1 and s_2+a_2 be elements of $S+I$

$\Rightarrow (s_1+a_1)-(s_2+a_2) = (s_1-s_2)+(a_1-a_2) \in S+I$

$(s_1+a_1)(s_2+a_2) = s_1s_2 + s_1a_2 + s_2a_1 + a_1a_2$
$\in S+I$

So $S+I$ is a subring

(2) $S \cap I \ni 0$ so non-empty. Let $s_1, s_2 \in S \cap I$ and let $a \in I$ and $s \in S$

Then $s_1 + s_2 \in S \cap I$, $s_1 - s_2 \in I$. Also ss_1 and $s_1 s \in S \cap I$

$s_2 s, ss_2 \in S \cap I$.

So for any $s \in S$ addition and multiplication of ideal is satisfied by $S \cap I$. So $S \cap I$ is ideal of S.

(3) $\phi : S \longrightarrow S+I/I$ This is a ring homomorphism
$s \longrightarrow s+I$

Let $s \in S$ and $a \in I$. Then $s + a + I = s + I \Rightarrow s+a+I \in img(\phi)$

So ϕ is surjection.

Let $h \in \ker(\phi)$, then, $h + I = I$ iff $h \in I$

and $h \in S$ as well $\Rightarrow h \in S \cap I$.

so by FIT we get, $S+I/I \cong S/(S \cap I)$

Third Isomorphism Theorem:-

Let R be a ring and let A, B be ideals of R with $B \subseteq A \subseteq R$. Then,

(1) The set A/B is an ideal of the quotient ring R/B

(2) $(R/B)/(A/B) \cong R/A$

Proof:- $A/B = \{ a+B : a \in A \}$

(1) Let, $a_1 + B, a_2 + B \in A/B$

$(a_1 + B) - (a_2 + B) = \underbrace{(a_1 - a_2)}_{\in A} + B \in A/B$

Let, $(r+B) \in R/B$ then, $(r+B)(a_1+B) = \underbrace{ra_1}_{\in A} + B \in A/B$

$\Rightarrow$ So A/B is an ideal of R/B

(2) Let, $\phi : R/B \longrightarrow R/A$
$r+B \longrightarrow r+A$

$r_1+B = r_2+B$

$\Rightarrow r_1 - r_2 \in B$ as $B \subseteq A$

$\Rightarrow r_1 - r_2 \in A$

$\phi(r_1 - r_2 + B) = r_1 - r_2 + A = A$

$\ker(\phi) = \{ r+B \mid \phi(r+B) = 0 \}$

$= \{ r+B \mid r+A = 0 \}$

$= \{ r+B \mid r \in A \}$

$= A/B$

By FIT we get, $(R/B)/(A/B) \cong R/A$

Correspondence Theorem:-

Let R be a ring and $I \trianglelefteq R$ be an ideal of R. The map $S \longrightarrow S/I$ defines a correspondance between the set of subrings of R containing I and the set of subrings of R/I. Similarly the map $J \rightarrow J/I$ gives a correspondace between the set of ideals of R containing I and the set of ideals of R/I.